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Compton scattering in strong magnetic fields in the laboratory frame

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Abstract

The Compton scattering cross-section in strong magnetic fields in the laboratory frame is derived, which gives Herold's result in the electron rest frame but cannot be recovered vice versa from the latter by relativistic transformations. For the scattering of a low-frequency photon with electrons, this cross-section has a simple form which can also be reduced to Herold's nonrelativistic limit but cannot be recovered either from the latter. Compared with previous works, this cross-section is directly applicable in astrophysical calculations for both relativistic and nonrelativistic cases and may lead to some revisions to past astrophysical calculations based upon Herold's expressions.

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Since Herold's study of the Compton scattering in strong magnetic fields in the electron rest frame (ERF) [1], his expression of cross-section has been widely used in astrophysical calculations, because it is believed that the inverse Compton scattering in strong magnetic fields near the surfaces of neutron stars is one of the possible mechanisms responsible for x-ray and γ-ray emissions. However, in actual calculations the cross-section in the laboratory frame (LF) is required. The difficulty of obtaining an LF version of Herold's cross-section in the ERF is due to the fact that the QED processes in an external magnetic field are relativistic invariant only in the direction of the field and, therefore, no exact relativistic transformations are available, as will be justified later in the text, to recover the full expression of cross section in the LF from Herold's expression. As a fact, up to now, most calculations were based on the relativistic transformation of Herold's nonrelativistic approximation, or of Dermer's expression [2, 3], similar to the treatment in the Thomson limit without external magnetic fields [4]. This kind of approximation is valid only for the scattering of a low-frequency photon with nonrelativistic electrons. In this paper the Compton scattering cross-section of a photon with relativistic electrons in ultra-strong magnetic fields is calculated but it is still in the ERF [5]. This paper aims at a derivation of an exact LF version of Herold's full expression of cross-section in the ERF.

Following the perturbation approach in the incoming interaction picture [6], a general expression is obtained in the LF for the cross-section of the Compton scattering in strong magnetic fields, which yields exactly Herold's result in the ERF, as it should be. This cross-section has a simple form for the scattering of a low-frequency photon with electrons, which can be reduced again to Herold's nonrelativistic limit. The advantage of this cross-section is that it can be used directly in astrophysical calculations for both relativistic and nonrelativistic cases.

1. Cross-section in laboratory frame

Charged particles on higher Landau levels have a very short lifetime τ due to the cyclotron radiation in strong magnetic fields [7], $\tau \leqslant 10^{-15}$ s. It is therefore reasonable to assume that electrons either before or after the magnetic scattering should be on the ground Landau level. That is to say an electron can only occupy a higher Landau level in its intermediate states. Taking into account the fact that the ground Landau level is not degenerate due to the spinor $u_{1,0}(x_1, k) = 0$ according to (A.4) in the appendix, the initial and final states of an electron–photon scattering system can be represented by

$$|i, t_i\rangle = c_0^+(\boldsymbol{p}_i, t_i)a_{\lambda_i}^+(\boldsymbol{k}_i, t_i)|0\rangle \tag{1}$$

$$|f, t_f\rangle = c_0^+(\boldsymbol{p}_f, t_f)a_{\lambda_f}^+(\boldsymbol{k}_f, t_f)|0\rangle \tag{2}$$

where, as shown in (A.12), $c_0^+ = c_{2,0}^+$ is the electron creation operator, a_λ^+ is the photon creation operator and $\boldsymbol{p}_{i(f)}$ denotes the incident (outgoing) electron momentum. We stress here that the operators in these two expressions are Heisenberg ones, not the free ones, meaning that interactions have been considered. The scattering matrix can be generally expressed by

$$S_{fi} = \lim_{t_i\to-\infty,\, t_f\to\infty} \langle 0|T[c_0(\boldsymbol{p}_f, t_f)a_{\lambda_f}(\boldsymbol{k}_f, t_f)c_0^+(\boldsymbol{p}_i, t_i)a_{\lambda_i}^+(\boldsymbol{k}_i, t_i)]|0\rangle. \tag{3}$$

Introducing the incoming interaction picture and making perturbation expansions [6], one then obtains under the Born approximation

$$\begin{aligned} S_{fi} = \lim_{t_i\to-\infty,\, t_f\to\infty} e^2 \int & \mathrm{d}^4x_1\, \mathrm{d}^4x_2\, \langle 0|Tc_0(\boldsymbol{p}_f, t_f)\bar{\psi}(x_1)|0\rangle\gamma_\mu\langle 0|T\psi(x_1)\bar{\psi}(x_2)|0\rangle\gamma_\nu \\ & \times[\langle 0|Ta_{\lambda_f}(\boldsymbol{k}_f, t_f)A_\mu(x_1)|0\rangle\langle 0|Ta_{\lambda_i}^+(\boldsymbol{k}_i, t_i)A_\nu(x_2)|0\rangle \\ & +\langle 0|Ta_{\lambda_f}(\boldsymbol{k}_f, t_f)A_\nu(x_2)|0\rangle\langle 0|Ta_{\lambda_i}^+(\boldsymbol{k}_i, t_i)A_\mu(x_1)|0\rangle] \\ & \times\langle 0|T\psi(x_2)c_0^+(\boldsymbol{p}_i, t_i)|0\rangle \end{aligned} \tag{4}$$

where operators are all the 'in' operators, i.e., the free ones, but, for simplification, we have omitted here the label 'in'. For example, the free Feynman propagator should be written as $\langle 0|T\psi_{in}(x)\bar{\psi}_{in}(y)|0\rangle$, but now it is denoted by $\langle 0|T\psi(x)\bar{\psi}(y)|0\rangle$, which is given in (A.17) and (A.18). According to (A.2) and (A.8), the incident (outgoing) electron momentum can be expressed as $\boldsymbol{p}_{i(f)} = (0, a_{i(f)}, p_{i(f)})$, meaning the incident (outgoing) electron momentum along the direction of the magnetic field (taken as the z direction) is $p_{i(f)}$ and the centre of the Landau orbit is $-\lambda^2 a_{i(f)}$, as shown in the harmonic oscillator wavefunction (A.6).

In this paper we adopt the normalization scheme in a box with a volume $V = L^3$. After a quite lengthy but straightforward calculation typical in QED and by neglecting an unimportant phase factor, the scattering amplitude can be derived as (see section 2 in appendix)

$$S_{fi} = \frac{(2\pi)^3}{VL^2}\frac{e^2}{2\sqrt{\omega_i\omega_f}}\left[\frac{(E_i+m)(E_f+m)}{4E_iE_f}\right]^{\frac{1}{2}}\exp\left[-\frac{\lambda^2}{4}(\omega_i^2\sin^2\theta_i+\omega_f^2\sin^2\theta_f)\right]$$
$$\times\exp\left[-\mathrm{i}\lambda^2 a_i(k_{ix}-k_{fx})-\mathrm{i}\frac{\lambda^2}{2}(k_{ix}k_{iy}+k_{fx}k_{fy})\right]X$$
$$\times\delta(E_i+\omega_i-E_f-\omega_f)\delta(p_i+k_i\cos\theta_i-p_f-k_f\cos\theta_f)$$
$$\times\delta(a_i+k_{iy}-a_f-k_{fy}) \tag{5}$$

where ω_i and ω_f are frequencies of the incident and scattered photons, E_i and E_f represent the energies of the incident and scattered electrons, respectively, θ_i (θ_f) denotes the angle between the incoming (outgoing) photon and the magnetic field and X is a rather complicated expression as follows:

$$X = \sum_{n=0}^{\infty}\frac{1}{n!}\left[\left(\frac{\lambda^2k_i^-k_f^+}{2}\right)^n\exp(\mathrm{i}\lambda^2k_{iy}k_{fx})\left(\frac{X_1}{(E_i+\omega_i)^2-E_{i,n+1}^2}+\frac{X_2}{(E_i+\omega_i)^2-E_{i,n}^2}\right)\right.$$
$$\left.+\left(\frac{\lambda^2k_i^+k_f^-}{2}\right)^n\exp(\mathrm{i}\lambda^2k_{ix}k_{fy})\left(\frac{X_1'}{(E_i-\omega_f)^2-E_{f,n+1}^2}+\frac{X_2'}{(E_i-\omega_f)^2-E_{f,n}^2}\right)\right] \tag{6}$$

in which $k_i^{\pm} = k_{ix}\pm\mathrm{i}k_{iy}$, $k_f^{\pm} = k_{fx}\pm\mathrm{i}k_{fy}$,

$$E_{i,n}^2 = m^2+(p_i+\omega_i\cos\theta_i)^2+2neB \tag{7}$$
$$E_{f,n}^2 = m^2+(p_i-\omega_f\cos\theta_f)^2+2neB \tag{8}$$

and $X_i, X_i', i = 1, 2$, are given by

$$X_1 = \left[\frac{(E_i+\omega_i+m)p_i(p_i+\omega_i\cos\theta_i-\omega_f\cos\theta_f)}{(E_i+m)(E_f+m)}+(E_i+\omega_i-m)\right]e_i^-e_f^+$$
$$+\left(\frac{p_i}{E_i+m}+\frac{p_i+\omega_i\cos\theta_i-\omega_f\cos\theta_f}{E_f+m}\right)$$
$$\times[k_f^+e_{fz}e_i^-+k_i^-e_{iz}e_f^+-(p_i+\omega_i\cos\theta_i)e_f^+e_i^-] \tag{9}$$

$$X_2 = \left[\frac{(E_i+\omega_i+m)p_i(p_i+\omega_i\cos\theta_i-\omega_f\cos\theta_f)}{(E_f+m)(E_i+m)}\right.$$
$$+\left(\frac{p_i}{E_i+m}+\frac{p_i+\omega_i\cos\theta_i-\omega_f\cos\theta_f}{E_f+m}\right)$$
$$\left.\times(p_i+\omega_i\cos\theta_i)+(E_i+\omega_i-m)\right]e_{fz}e_{iz} \tag{10}$$

$$X_1' = \left[\frac{(E_i-\omega_f+m)p_i(p_i+\omega_i\cos\theta_i-\omega_f\cos\theta_f)}{(E_f+m)(E_i+m)}+(E_i-\omega_f-m)\right]e_i^+e_f^-$$
$$-\left(\frac{p_i}{E_i+m}+\frac{p_i+\omega_i\cos\theta_i-\omega_f\cos\theta_f}{E_f+m}\right)$$
$$\times[k_i^+e_{iz}e_f^-+k_f^-e_{fz}e_i^++(p_i-\omega_f\cos\theta_f)e_i^+e_f^-] \tag{11}$$

$$X_2' = \left[\frac{(E_i-\omega_f+m)p_i(p_i+\omega_i\cos\theta_i-\omega_f\cos\theta_f)}{(E_f+m)(E_i+m)}+(E_i-\omega_f-m)\right.$$
$$\left.+\left(\frac{p_i}{E_i+m}+\frac{p_i+\omega_i\cos\theta_i-\omega_f\cos\theta_f}{E_f+m}\right)(p_i-\omega_f\cos\theta_f)\right]e_{iz}e_{fz} \tag{12}$$

where e_i and e_f are polarizations of the incident and scattered photons and $e_i^{\pm} = e_{ix} \pm \mathrm{i}e_{iy}$, $e_f^{\pm} = e_{fx} \pm \mathrm{i}e_{fy}$. The conservation of energy and momentum along the z direction, i.e. $E_f = E_i + \omega_i - \omega_f$ and $p_f + k_f \cos\theta_f = p_i + k_i \cos\theta_i$, leads to

$$\omega_f = \frac{1}{\sin^2\theta_f}\{E_i - p_i\cos\theta_f + \omega_i(1 - \cos\theta_i\cos\theta_f) - [(E_i - p_i\cos\theta_f)^2 + 2\omega_i(E_i\cos\theta_f - p_i)(\cos\theta_f - \cos\theta_i) + \omega_i^2(\cos\theta_f - \cos\theta_i)^2]^{\frac{1}{2}}\}. \tag{13}$$

From the scattering amplitude (5) and following the routine procedure, i.e. summing over the final states of the scattered photon and electron, $\boldsymbol{k}_f,\ a_f,\ p_f$, to obtain the rate of scattering probability which is then divided by the relative incident flux density, $(E_i - p_i\cos\theta_i)/VE_i$, the differential cross-section can thus be obtained:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega_f} = \frac{r_0^2}{4}\frac{\omega_f}{\omega_i}\frac{m^2}{(E_i+m)(E_f+m)}\frac{\exp[-\frac{\lambda^2}{2}(\omega_i^2\sin^2\theta_i + \omega_f^2\sin^2\theta_f)]|Y|^2}{(E_i - p_i\cos\theta_i)[E_f - (p_i + \omega_i\cos\theta_i - \omega_f\cos\theta_f)\cos\theta_f]} \tag{14}$$

in which r_0 is the classical electron radius and $Y = Y_1 + Y_2$, which are given by

$$Y_1 = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\lambda^2\omega_i\omega_f\sin\theta_i\sin\theta_f}{2}\mathrm{e}^{-\mathrm{i}(\phi_i-\phi_f)}\right)^n \exp(\mathrm{i}\lambda^2\omega_i\omega_f\sin\theta_i\sin\theta_f\cos\phi_f\sin\phi_i) \times\left[\frac{[A - B(p_i + \omega_i\cos\theta_i)]e_f^+e_i^- + B(k_f^+e_{fz}e_i^- + k_i^-e_{iz}e_f^+]}{(E_i+\omega_i)^2 - E_{i,n+1}^2} + \frac{[A + B(p_i + \omega_i\cos\theta_i)]e_{fz}e_{iz}}{(E_i+\omega_i)^2 - E_{i,n}^2}\right] \tag{15}$$

$$Y_2 = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\lambda^2\omega_i\omega_f\sin\theta_i\sin\theta_f}{2}\mathrm{e}^{\mathrm{i}(\phi_i-\phi_f)}\right)^n \exp(\mathrm{i}\lambda^2\omega_i\omega_f\sin\theta_i\sin\theta_f\sin\phi_f\cos\phi_i) \times\left[\frac{[A' - B(p_i - \omega_f\cos\theta_f)]e_f^-e_i^+ - B(k_f^-e_{fz}e_i^+ + k_i^+e_{iz}e_f^-)}{(E_i-\omega_f)^2 - E_{f,n+1}^2} + \frac{[A' + B(p_i - \omega_f\cos\theta_f)]e_{fz}e_{iz}}{(E_i-\omega_f)^2 - E_{f,n}^2}\right]. \tag{16}$$

In the above two expressions the coefficients $A,\ A',\ B$ are defined by

$$A = (E_i + \omega_i + m)p_ip_f + (E_i + \omega_i - m)(E_i + m)(E_f + m) \tag{17}$$

$$A' = (E_i - \omega_f + m)p_ip_f + (E_i - \omega_f - m)(E_i + m)(E_f + m) \tag{18}$$

$$B = p_i(E_f + m) + p_f(E_i + m) \tag{19}$$

respectively. In the ERF $p_i = 0$, (14) is reduced to Herold's expression, as expected. However the expression (14) cannot be derived vice versa by relativistic transformations from the latter. For example, the photon frequencies in (14) are not all the Doppler ones as they would be according to relativistic transformation rules, and the terms in (14) multiplied by p_i like those in (17)–(19) cannot be recovered, since one obtains nothing from zero by relativistic transformations.

To simplify calculations, we choose the coordinate system with $\phi_i = 0$. Denoting $\phi_f = \phi$, we can write

$$\boldsymbol{k}_i = \omega_i(\sin\theta_i,\ 0,\ \cos\theta_i) \qquad \boldsymbol{k}_f = \omega_f(\sin\theta_f\cos\phi,\ \sin\theta_f\sin\phi,\ \cos\phi). \tag{20}$$

Taking into account the fact that photons have only two transversal polarizations, the polarizations of the incident and scattered photons can be chosen as

$$e_i^{(1)} = (-\cos\theta_i,\ \ 0,\ \ \sin\theta_i) \qquad e_i^{(2)} = (0,\ \ -1,\ \ 0) \tag{21}$$

$$e_f^{(1)} = (-\cos\theta_f\cos\phi,\ \ -\cos\theta_f\sin\phi,\ \ \sin\theta_f) \qquad e_f^{(2)} = (\sin\phi,\ \ -\cos\phi,\ \ 0). \tag{22}$$

The above choice is not unique but is convenient for calculations. Now we define reduced quantities

$$\Delta_i = \frac{\omega_i}{m} \qquad \Delta_f = \frac{\omega_f}{m} \qquad \Delta_0 = \frac{\omega_0}{m} \tag{23}$$

where $\omega_0 = eB/m$ is the cyclotron frequency. We denote the reduced Doppler frequencies by

$$\Delta_{ir} = \gamma(1-\beta\cos\theta_i)\Delta_i \qquad \Delta_{fr} = \gamma(1-\beta\cos\theta_f)\Delta_f. \tag{24}$$

Averaging over the polarizations of incident photons and summing over those ones of scattered photons, the total differential cross-section is obtained:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega_f} = \frac{r_0^2}{8}\frac{\Delta_f}{\Delta_{ir}(1+\gamma)(1+\gamma+\Delta_i-\Delta_f)} \times \frac{\exp[-\frac{B_c}{2B}(\Delta_f^2\sin^2\theta_f+\Delta_i^2\sin^2\theta_i)]|Y'|^2}{[\gamma(1-\beta\cos\theta_f)+\Delta_i(1-\cos\theta_i\cos\theta_f)-\Delta_f\sin^2\theta_f]} \tag{25}$$

in which $B_c = \frac{m^2}{e} \approx 4.414\times10^9$ T is the critical magnetic field and $|Y'|^2$ is given by

$$|Y'|^2 = |Y'(1_i\to1_f)|^2 + |Y'(1_i\to2_f)|^2 + |Y'(2_i\to1_f)|^2 + |Y'(2_i\to2_f)|^2 \tag{26}$$

where $\lambda_i \to \lambda_f,\ \lambda_{i(f)} = 1_{i(f)},\ 2_{i(f)}$, represents the scattering of a photon from the polarization λ_i to λ_f, and

$$\begin{aligned} Y'(1_i\to1_f) = {} & [(A_-\cos\theta_f - B_1)\cos\theta_i - B_2\cos\theta_f]\sum_{n=0}^{\infty}\frac{1}{n!}\zeta^n S_{i,n+1}\mathrm{e}^{[\mathrm{i}(n+1)\phi]} \\ & -[(A'_-\cos\theta_f + B_1)\cos\theta_i + B_2\cos\theta_f]\sum_{n=0}^{\infty}\frac{1}{n!}\zeta^n S_{f,n+1}\mathrm{e}^{-\mathrm{i}[(n+1)\phi-\eta]} \\ & +\sin\theta_i\sin\theta_f\left[A_+\sum_{n=0}^{\infty}\frac{1}{n!}\zeta^n S_{i,n}\mathrm{e}^{\mathrm{i}n\phi} - A'_+\sum_{n=0}^{\infty}\frac{1}{n!}\zeta^n S_{f,n}\mathrm{e}^{-\mathrm{i}(n\phi-\eta)}\right] \end{aligned} \tag{27}$$

where

$$A_\pm = a \pm b(\beta\gamma + \Delta_i\cos\theta_i) \qquad A'_\pm = a' \pm b(\beta\gamma - \Delta_f\cos\theta_f) \tag{28}$$

$$B_1 = b\Delta_f\sin^2\theta_f \qquad B_2 = b\Delta_i\sin^2\theta_i \tag{29}$$

in which

$$a = \beta\gamma(1+\gamma+\Delta_i)(\beta\gamma+\Delta_i\cos\theta_i-\Delta_f\cos\theta_f) + (\gamma-1+\Delta_i)(1+\gamma)(1+\gamma+\Delta_i-\Delta_f)$$
$$a' = \beta\gamma(1+\gamma-\Delta_f)(\beta\gamma+\Delta_i\cos\theta_i-\Delta_f\cos\theta_f) + (\gamma-1-\Delta_f)(1+\gamma)(1+\gamma+\Delta_i-\Delta_f)$$
$$b = \beta\gamma(1+\gamma+\Delta_i-\Delta_f) + (\beta\gamma+\Delta_i\cos\theta_i-\Delta_f\cos\theta_f)(1+\gamma)$$

and

$$\eta = \zeta\sin\phi \qquad \zeta = \frac{B_c}{2B}\Delta_i\Delta_f\sin\theta_i\sin\theta_f$$
$$S_{i,n} = \frac{1}{2(\Delta_{ir}-n\Delta_0)+\Delta_i^2\sin^2\theta_i} \qquad S_{f,n} = \frac{1}{2(\Delta_{fr}+n\Delta_0)-\Delta_f^2\sin^2\theta_f}.$$

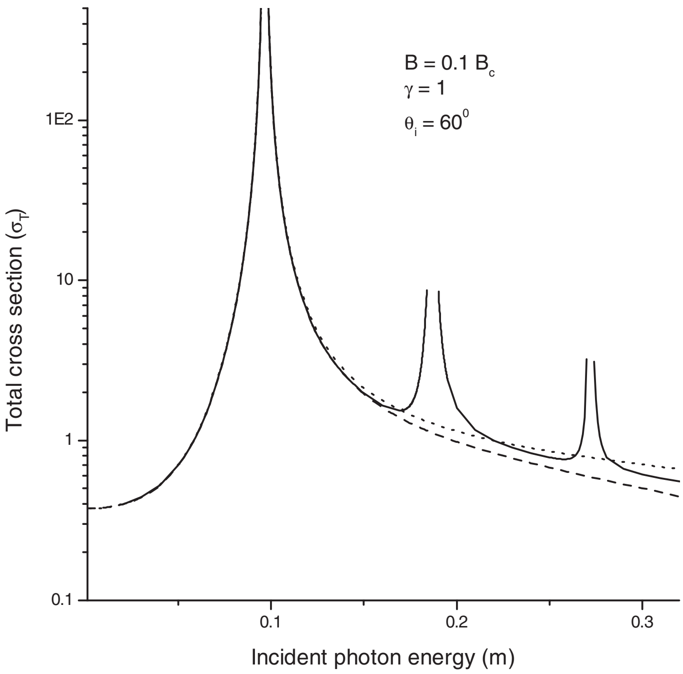

Figure 1. The total cross-section in the ERF ($\gamma = 1$). The incident photon energy is in units of the electron rest energy (m) and the magnetic field B is in units of B_c. The total cross-section is in Thomson units (σ_T). The solid curve is plotted according to the full expression (25) where the exact summations in (27), (30), (31) and (32) have been performed numerically. It is seen that apparent resonances occur at higher Landau levels. The dotted curve results from the $n = 0$ terms (i.e., summing up to the first Landau level) in (27), (30), (31) and (32), which is valid for $\Delta_i \ll 1$. The dashed curve corresponds to (34) and (35) together with the simplified expression (41). Since the $n = 0$ approximation takes into account only the first Landau level, the resonances at higher Landau levels are absent from the dotted (dashed) curve.

Other Y' are given by

$$Y'(1_i \to 2_f) = \mathrm{i}(A_- \cos\theta_i - B_2) \sum_{n=0}^{\infty} \frac{1}{n!} \zeta^n S_{i,n+1} \mathrm{e}^{\mathrm{i}(n+1)\phi} + \mathrm{i}(A'_- \cos\theta_i + B_2) \sum_{n=0}^{\infty} \frac{1}{n!} \zeta^n S_{f,n+1} \mathrm{e}^{-\mathrm{i}(n+1)\phi + \mathrm{i}\eta} \tag{30}$$

$$Y'(2_i \to 1_f) = -\mathrm{i}(A_- \cos\theta_f - B_1) \sum_{n=0}^{\infty} \frac{1}{n!} \zeta^n S_{i,n+1} \mathrm{e}^{\mathrm{i}(n+1)\phi} - \mathrm{i}(A'_- \cos\theta_f + B_1) \sum_{n=0}^{\infty} \frac{1}{n!} \zeta^n S_{f,n+1} \mathrm{e}^{-\mathrm{i}(n+1)\phi + \mathrm{i}\eta} \tag{31}$$

$$Y'(2_i \to 2_f) = A_- \sum_{n=0}^{\infty} \frac{1}{n!} \zeta^n S_{i,n+1} \mathrm{e}^{\mathrm{i}(n+1)\phi} - A'_- \sum_{n=0}^{\infty} \frac{1}{n!} \zeta^n S_{f,n+1} \mathrm{e}^{-\mathrm{i}(n+1)\phi + \mathrm{i}\eta}. \tag{32}$$

Now we are ready to carry out numerical calculations and the results are shown by solid curves in figures 1–3 in which $B = 0.1B_c$. The solid line in figure 1 is plotted for comparison with the corresponding one of Herold and it is clear that they are identical. Figures 2 and 3 consider the

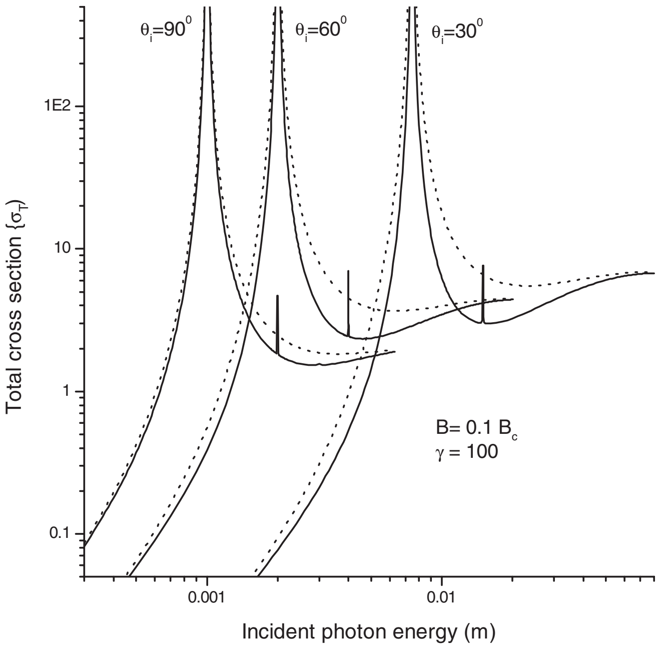

Figure 2. Total cross-sections for different incident photon angles when the incident electron energy is given ($\gamma = 100$). It is seen from solid curves, as compared with figure 1, that even the resonance at the second Landau level is very weak in the LF. The smaller the incident photon angle, the higher the incident photon energy required to reach a resonant scattering. The dotted curves are plotted according the $n = 0$ approximation where the resonances at higher Landau levels are absent. The deviation of dotted curves from solid ones for higher incident photon energies can be viewed as a compensation for the absence of higher-order resonances in the approximation.

scattering of low-frequency photons with high-energy electrons. The solid curves in figures 2 and 3 show that the second-order resonance is very weak as compared with the first-order one. This is understandable for the reason that the transition probability of a relativistic electron to higher Landau levels by absorbing an incident photon should be obviously less then that to the first Landau level. However this is not obvious in the ERF, because apparent resonance occurs for higher Landau levels in the ERF as shown in figure 1. Figure 2 is to show the influence of incident angles of photons over the cross-section for a given energy, while figure 3 considers the effect of the energies of incident electrons on the cross-section for a given incident angle. It is seen in figure 2 that a change in incident angles leads to a shift of resonance: the smaller the incident angle, the higher the incident photon energy required to reach a resonant scattering. This is understandable by physical intuitions that for a photon to be resonantly scattered by a high-energy electron with nearly the speed of light its energy should be higher in the case of smaller incident angles.

In astrophysics one is more interested in the inverse Compton scattering of a low-energy photon gas with high-energy electrons in a strong magnetic field. When $\Delta_i \ll 1$ the summations in (27) and (30)–(32) converge rapidly and the $n = 0$ terms dominate the contribution, which is verified by numerical results shown by dotted curves in figures 1–3. It is seen that the deviation of dotted curves from solid ones for higher incident photon energies can be considered as a compensation for the absence of higher-order resonances in the approximation.

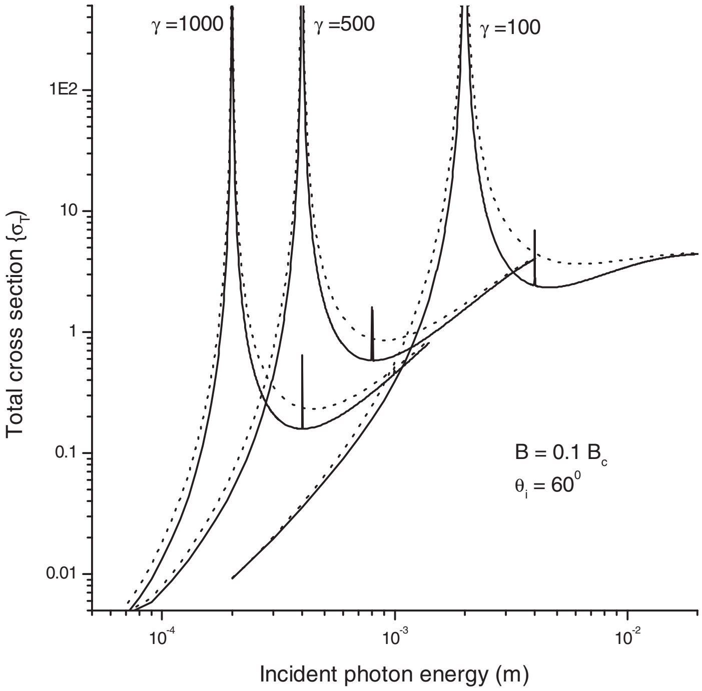

Figure 3. Total cross-sections, with different incident photon energies when the incident photon angle is fixed ($\theta_i = 60°$). It is seen that the higher the electron energy, the lower the photon energy required to reach a resonant scattering. The solid curves originate from the full expression (25), while the dotted curves correspond to the $n = 0$ approximation and have the same meaning as in figure 2.

For further simplification, we integrate over the azimuthal angle ϕ and introduce the Bessel functions

$$J_n(-\xi) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi \cos[n\phi - \xi \sin(\phi)] \qquad n = 0, 1, 2 \tag{33}$$

then the differential cross-section (25), summing up to $n = 0$, can be reduced to

$$\mathrm{d}\sigma = \sigma(\Delta_i, \theta_i, \gamma, \theta_f)\sin\theta_f \,\mathrm{d}\theta_f \tag{34}$$

where

$$\sigma(\Delta_i, \theta_i, \gamma, \theta_f) = \frac{\pi r_0^2}{4}\frac{\Delta_f}{\Delta_{ir}(\gamma+1)(1+\gamma+\Delta_i-\Delta_f)} \times \frac{\exp(-\frac{B_c}{2B}\Delta_f^2\sin^2\theta_f)Y_r}{[\gamma(1-\beta\cos\theta_f)+\Delta_i(1-\cos\theta_i\cos\theta_f)-\Delta_f\sin^2\theta_f]} \tag{35}$$

with Y_r given by

$$\begin{aligned} Y_r = {} & C_1 S_{i,1}^2 + C_2 S_{f,1}^2 + [(A_+ S_{i,0} - A'_+ S_{f,0})^2 + 2(1 - J_0(\zeta))A_+ A'_+ S_{i,0} S_{f,0}](\sin\theta_i \sin\theta_f)^2 \\ & +2[(A_- \cos\theta_i - B_2)(A'_- \cos\theta_i + B_2) + (A_- \cos\theta_f - B_1) \\ & \times(A'_- \cos\theta_f + B_1) - D_1 D_2 \\ & - A_- A'_-] J_2(-\xi) S_{i,1} S_{f,1} - 2[D_1 A'_+ S_{i,1} S_{f,0} + D_2 A_+ S_{i,0} S_{f,1}] \\ & \times \sin\theta_i \sin\theta_f J_1(-\xi) \end{aligned} \tag{36}$$

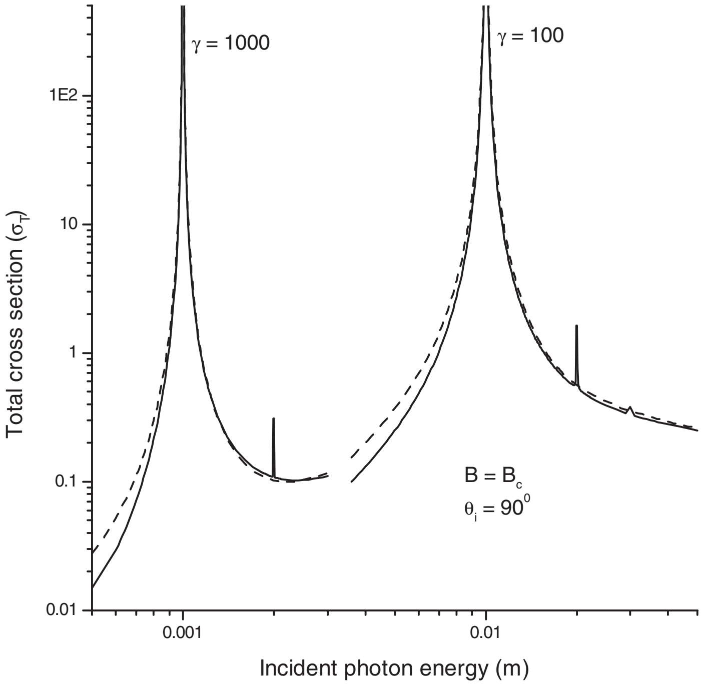

Figure 4. The dashed curves are plotted according to (34), (35) and the simplified expression (41) and are compared with solid curves according to the full expression (25), where $B = B_c$, $\theta_i = 90°$. The dashed curves, as well as the dashed curve in figure 1, show that the simplified form is satisfactory under various physical conditions.

in which

$$D_1 = (A_- \cos\theta_f - B_1)\cos\theta_i - B_2\cos\theta_f \tag{37}$$

$$D_2 = (A'_- \cos\theta_f + B_1)\cos\theta_i + B_2\cos\theta_f \tag{38}$$

and

$$C_1 = D_1^2 + (A_- \cos\theta_i - B_2)^2 + (A_- \cos\theta_f - B_1)^2 + A_-^2 \tag{39}$$

$$C_2 = D_2^2 + (A'_- \cos\theta_i + B_2)^2 + (A'_- \cos\theta_f + B_1)^2 + A'^2_-. \tag{40}$$

Then expression (36) can be simplified to

$$Y_r = C_1 S_{i,1}^2 + C_2 S_{f,1}^2 + [(A_+ S_{i,0} - A'_+ S_{f,0})]^2 (\sin\theta_i \sin\theta_f)^2$$
$$+2(1 - J_0(\zeta))A_+ A'_+ S_{i,0} S_{f,0}](\sin\theta_i \sin\theta_f)^2 \tag{41}$$

by noting that $J_1(-\xi)$ and $J_2(-\xi)$ are negligible for $\Delta_i \ll 1$. To justify this, numerical calculations have been performed; these are shown by dashed curves in figures 1 and 4. We find that the last term containing $J_0(\xi)$ in (41) is important for this approximation in the sector of higher incident photon energies. It is worth pointing out that the approximation (41) can also be justified by considering the nonrelativistic limit in the ERF ($\gamma = 1$, or $\beta = 0$). Under the Thomson limit ($\Delta_f \simeq \Delta_i \ll 1$) it is easy to see $A_- = -A'_- \simeq 4\Delta_i$, $B_1 = B_2 \simeq 0$, $A_+ = -A'_+ \simeq 4\Delta_i$ and $J_0(\xi) \simeq 1$; (35) becomes

$$\frac{\sigma(\Delta_i,\theta_i,\theta_f)}{\pi r_0^2} = \sin^2\theta_i \sin^2\theta_f + \frac{1}{4}(1+\cos^2\theta_i)(1+\cos^2\theta_f)\left[\frac{\Delta_i^2}{(\Delta_i-\Delta_0)^2} + \frac{\Delta_i^2}{(\Delta_i+\Delta_0)^2}\right] \tag{42}$$

which is just Herold's nonrelativistic result, a well known expression widely used in astrophysics. However (35) cannot be recovered vice versa from (42) by relativistic transformations.

To conclude, we have derived the magnetic Compton scattering cross-section and its simplified form in the LF, which can be reduced to Herold's results in the ERF but cannot be recovered vice versa from Herold's expressions by relativistic transformations. This implies that this cross-section may lead to some revisions to past astrophysical calculations based upon (42), for example, the number and power spectra of scattered photons resulting from the magnetic inverse Compton scattering of a photon gas with a relativistic electron beam which are detectable for observers. In fact, based upon the simplified expressions (34), (35) and (41) we have calculated the spectrum of the scattered photon number (i.e. the rate of the scattered photon number per unit scattered photon energy and per unit volume versus the scattered photon energy) resulting from the scattering of a thermal photon gas with a relativistic electron beam, which is supposed to happen at the surface of a neuton star. It is found that there exists a high-energy tail beyond each resonant scattering due to the presence of the last term in (41), which is absent from the work of Daugherty and Harding (figure 9) where (42) was used in Monte Carlo simulations [8]. As a consequence, the highest scattered photon energy we calculated is much higher than that obtained by Daugherty and Harding. The details will be given in a separate paper.

Appendix

A.1. Derivation of the magnetic Feynman propagator

We begin with the Dirac equation

$$[\mathrm{i}\,\not{\partial} - e\,\not{A}(x) - m]\psi(x,t) = 0 \tag{A.1}$$

in which $e > 0$ is assumed and the asymmetry gauge is taken, i.e. $\boldsymbol{A}(x) = (0,\ Bx_1,\ 0)$. The solution of (A.1) with positive energy is

$$\psi_{s,n}^{(+)}(\boldsymbol{x},t) = Nu_{s,n}(x_1,\boldsymbol{p})\exp[-\mathrm{i}E_n(p_3)t + \mathrm{i}p_2x_2 + \mathrm{i}p_3x_3] \qquad s = 1, 2 \tag{A.2}$$

where E_n indicates the Landau energy of an electron and N is a normalization constant

$$E_n(p_3) = (m^2 + p_3^2 + 2neB)^{\frac{1}{2}} \qquad N = \left[\frac{E_n(p_3)+m}{2E_n(p_3)}\right]^{\frac{1}{2}}. \tag{A.3}$$

The spinors $u_{1,n}(x_1,\boldsymbol{p})$, $u_{2,n}(x_1,\boldsymbol{p})$ are given by

$$u_{1,n}(x_1,\boldsymbol{p}) = \begin{bmatrix} I_{n-1}(x_1,\ p_2) \\ 0 \\ \frac{p_3}{E_n(p_3)+m} I_{n-1}(x_1,\ p_2) \\ \frac{\mathrm{i}\sqrt{2neB}}{E_n(p_3)+m} I_n(x_1,\ p_2) \end{bmatrix} \tag{A.4}$$

$$u_{2,n}(x_1,\boldsymbol{p}) = \begin{bmatrix} 0 \\ I_n(x_1,\ p_2) \\ \frac{-\mathrm{i}\sqrt{2neB}}{E_n(p_3)+m} I_{n-1}(x_1,\ p_2) \\ \frac{-p_3}{E_n(p_3)+m} I_n(x_1,\ p_2) \end{bmatrix} \tag{A.5}$$

in which $I_n(x_1, p_2)$ is the harmonic oscillator wavefunction

$$I_n(x_1, p_2) = (\lambda\sqrt{\pi}2^n n!)^{-\frac{1}{2}} \exp\left[-\frac{1}{2}\left(\frac{x_1}{\lambda} + \lambda p_2\right)^2\right] H_n\left(\frac{x_1}{\lambda} + \lambda p_2\right) \tag{A.6}$$

obeying the orthogonal condition

$$\int \mathrm{d}x_1\, I_n(x_1, p_2) I_m(x_1, p_2) = \delta_{nm} \tag{A.7}$$

where $\lambda^{-1} = \sqrt{eB}$. The solution of (A.1) with negative energy is

$$\psi_{s,n}^{(-)}(x) = N v_{s,n}(x_1, \boldsymbol{p}) \exp[\mathrm{i}E_n(p_3)t + \mathrm{i}p_2x_2 + \mathrm{i}p_3x_3] \qquad s = 1, 2 \tag{A.8}$$

with the spinors $v_{1,n}(x_1, \boldsymbol{p})$, $v_{2,n}(x_1, \boldsymbol{p})$ given by

$$v_{1,n}(x_1, \boldsymbol{p}) = \begin{bmatrix} \frac{-p_3}{E_n(p_3)+m} I_{n-1}(x_1, p_2) \\ \frac{-\mathrm{i}\sqrt{2neB}}{E_n(p_3)+m} I_n(x_1, p_2) \\ I_{n-1}(x_1, p_2) \\ 0 \end{bmatrix} \tag{A.9}$$

$$v_{2,n}(x_1, \boldsymbol{p}) = \begin{bmatrix} \frac{\mathrm{i}\sqrt{2neB}}{E_n(p_3)+m} I_{n-1}(x_1, p_2) \\ \frac{p_3}{E_n(p_3)+m} I_n(x_1, p_2) \\ 0 \\ I_n(x_1, p_2) \end{bmatrix}. \tag{A.10}$$

It is obvious that the spinors $u_{s,n}(x_1, \boldsymbol{p})$, $v_{s,n}(x_1, \boldsymbol{p}), s = 1, 2$ form a complete and orthogonalized basis in the four-dimensional vector space. Therefore the Dirac field operators can be expanded [9] as

$$\psi(x) = \sum_{n=0}^{\infty} \sum_{s=1}^{2} \frac{1}{L} \sum_{p_2,\, p_3} [c_{s,n}(\boldsymbol{p}, t) u_{s,n}(x_1, \boldsymbol{p}) + d_{s,n}^{+}(\boldsymbol{p}, t) v_{s,n}(x_1, \boldsymbol{p})] \exp(\mathrm{i}p_2x_2 + \mathrm{i}p_3x_3) \tag{A.11}$$

$$\psi^{+}(x) = \sum_{n=0}^{\infty} \sum_{s=1}^{2} \frac{1}{L} \sum_{p_2,\, p_3} [c_{s,n}^{+}(\boldsymbol{p}, t) u_{s,n}^{+}(x_1, \boldsymbol{p}) + d_{s,n}(\boldsymbol{p}, t) v_{s,n}^{+}(x_1, \boldsymbol{p})] \exp(-\mathrm{i}p_2x_2 - \mathrm{i}p_3x_3). \tag{A.12}$$

From the commutation relation

$$\{\psi_\alpha(\boldsymbol{x}, t), \psi_\beta^{+}(\boldsymbol{y}, t)\} = \delta_{\alpha\beta}\delta^3(\mathbf{x} - \boldsymbol{y}) \tag{A.13}$$

the commutation relations between the annihilation and destruction operators can be derived as

$$\{c_{s,n}(\boldsymbol{p}_1, t), c_{r,m}^{+}(\boldsymbol{p}_2, t)\} = \delta_{sr}\delta_{nm}\delta_{\boldsymbol{p}_1\boldsymbol{p}_2} \tag{A.14}$$

$$\{d_{sn}(\boldsymbol{p}_1, t), d_{rm}^{+}(\boldsymbol{p}_2, t)\} = \delta_{sr}\delta_{nm}\delta_{\boldsymbol{p}_1\boldsymbol{p}_2}. \tag{A.15}$$

The Feynman propagator of an electron is defined by

$$S_F(x, y) = -\mathrm{i}\langle 0|T\psi(x)\bar{\psi}(y)|0\rangle. \tag{A.16}$$

The substitution of (12) and (13) into above expression leads, after some algebra, to

$$S_F(x, y) = \frac{1}{L^2} \sum_{p_2\, p_3} \int \frac{\mathrm{d}\omega}{2\pi} \sum_{n=0}^{\infty} \frac{S_n(x_1, y_1, \boldsymbol{p})}{\omega^2 - E_n^2(p_3) + \mathrm{i}\varepsilon} \times \exp[-\mathrm{i}\omega(t_x - t_y) + \mathrm{i}p_2(x_2 - y_2) + \mathrm{i}p_3(x_3 - y_3)] \tag{A.17}$$

where $S_n(x_1, y_1, \boldsymbol{p}) \equiv S_n(x_1, y_1, p_2, p_3, \omega)$ is defined by

$$S_n(x_1, y_1, \boldsymbol{p}) = \begin{bmatrix} (\omega+m)I_{n-1}I_{n-1} & 0 & -p_3 I_{n-1}I_{n-1} & \mathrm{i}\sqrt{2neB}I_{n-1}I_n \\ 0 & (\omega+m)I_nI_n & -\mathrm{i}\sqrt{2neB}I_nI_{n-1} & p_3I_nI_n \\ p_3I_{n-1}I_{n-1} & -\mathrm{i}\sqrt{2neB}I_{n-1}I_n & -(\omega-m)I_{n-1}I_{n-1} & 0 \\ \mathrm{i}\sqrt{2neB}I_nI_{n-1} & -p_3I_nI_n & 0 & -(\omega-m)I_nI_n \end{bmatrix} \tag{A.18}$$

in which I_mI_n is the abbreviation for $I_m(x_1, p_2)I_n(y_1, p_2)$.

A.2. Calculation of the scattering amplitude

In this section we give a detailed calculation of the scattering amplitude (5). Taking into account that $t_f \to \infty$, it is easy to see from (A.12)

$$\begin{aligned}\langle 0|Tc_0(\boldsymbol{p}_f, t_f)\bar{\psi}(\boldsymbol{r}_1, t)|0\rangle &= \langle 0\{c_0(\boldsymbol{p}_f, t_f), \bar{\psi}^{(-)}(\boldsymbol{r}_1, t)\}\rangle \\ &= \frac{1}{L}\bar{u}_0(x_1, \boldsymbol{p}_f)\exp(-\mathrm{i}E_ft_f - \mathrm{i}p_fz_1 - \mathrm{i}a_fy_1 + \mathrm{i}E_ft_1)\end{aligned} \tag{A.19}$$

where we have set $u_0 = u_{20}$, because $u_{10} = 0$ ($I_{-1}(x_1, p_2) = 0$ by definition). Similarly we have

$$\langle 0|Ta_{\lambda_f}(\boldsymbol{k}_f, t_f)A_\mu(\boldsymbol{r}_1, t)|0\rangle = \frac{1}{\sqrt{V}}\frac{1}{\sqrt{2\omega_f}}e_\mu^{(\lambda_f)}\exp(-\mathrm{i}\omega_ft_f + \mathrm{i}\omega_ft_1 - \mathrm{i}\boldsymbol{k}_f\cdot\boldsymbol{r}_1) \tag{A.20}$$

$$\langle 0|T\psi(\boldsymbol{r}_2, t)c_0^+(\boldsymbol{p}_i, t_i)|0\rangle = \frac{1}{L}u_0(x_2, \boldsymbol{p}_i)\exp(\mathrm{i}E_it_i + \mathrm{i}p_iz_2 + \mathrm{i}a_iy_2 - \mathrm{i}E_it_2) \tag{A.21}$$

$$\langle 0|Ta_{\lambda_i}^+(\boldsymbol{k}_i, t_i)A_\nu(\boldsymbol{r}_2, t)|0\rangle = \frac{1}{\sqrt{V}}\frac{1}{\sqrt{2\omega_i}}e_\nu^{(\lambda_i)}\exp(\mathrm{i}\omega_it_i - \mathrm{i}\omega_it_2 + \mathrm{i}\boldsymbol{k}_i\cdot\boldsymbol{r}_2) \tag{A.22}$$

where $e^{(\lambda_i)} = e_i$ ($e^{(\lambda_f)} = e_f$) denotes the polarization of the incident (outgoing) photon. Substituting the Feynman propagator (A.17) and the expressions (A.18) and (A.19) into (4) we obtain after some algebra

$$\begin{aligned}S_{fi} = \frac{(2\pi)^5}{L^2V}\frac{e^2}{2\sqrt{\omega_i\omega_f}}\sum_{n=0}^{\infty}\frac{1}{L^2}\sum_{q_yq_z}\int \mathrm{d}^4x_1\,\mathrm{d}^4x_2\int \mathrm{d}q_0\,\bar{u}_0(x_1, \boldsymbol{p}_f) \\ \times\Bigg[\hat{e}_f\frac{S_n(x_1, x_2, \boldsymbol{q})}{(E_i+\omega_i)^2 - E_n^2(q_z) + \mathrm{i}\epsilon}\hat{e}_i\,\delta(\omega_f + E_f - q_0)\delta(q_0 - \omega_i - E_i) \\ \times\delta(k_{iy} + a_i - q_y) \\ \times\delta(q_y - k_{fy} - a_f)\delta(q_z - k_f\cos\theta_f - p_f)\delta(k_i\cos\theta_i + p_i - q_z) \\ \times\exp(-\mathrm{i}k_{fx}x_1 + \mathrm{i}k_{ix}x_2) \\ +\hat{e}_i\frac{S_n(x_1, x_2, \boldsymbol{q})}{(E_i - \omega_f)^2 - E_n^2(q_z) + \mathrm{i}\epsilon}\hat{e}_f \\ \times\delta(\omega_f - E_i + q_0)\delta(E_f - \omega_i - q_0)\delta(k_{iy} - a_f + q_y) \\ \times\delta(a_i - k_{fy} - q_y)\delta(q_z + k_i\cos\theta_i - p_f)\delta(p_i - k_f\cos\theta_f - q_z) \\ \times\exp(\mathrm{i}k_{ix}x_1 - \mathrm{i}k_{fx}x_2)\Bigg] \\ \times u_o(x_2, \boldsymbol{p}_i)\lim_{t_i\to-\infty,\, t_f\to\infty}\exp[-\mathrm{i}(E_f+\omega_f)t_f + \mathrm{i}(E_i-\omega_i)t_i]\end{aligned} \tag{A.23}$$

where $\hat{e}_f = e_\mu^{(\lambda_f)}\gamma_\mu$, $\hat{e}_i = e_\mu^{(\lambda_i)}\gamma_\mu$ The phase factor at the end of (A.23) can be ignored, since the cross-section $\sim|S_{fi}|^2$. The spinors $\bar{u}_0(x_1, \boldsymbol{p}_f)$, $u_0(x_2, \boldsymbol{p}_i)$ can be expressed as

$$\bar{u}_0(x_1, \boldsymbol{p}_f) = \left[\frac{E_f + m}{2E_f}\right]^{\frac{1}{2}} I_0(x_1, a_f)\bar{u}_f(p_f) \tag{A.24}$$

$$u_0^T(x_2,\boldsymbol{p}_i) = \left[\frac{E_i+m}{2E_i}\right]^{\frac{1}{2}} I_0(x_2,a_i)\bar{u}_i^T(p_i) \tag{A.25}$$

where

$$\bar{u}_f(p_f) = \begin{pmatrix} 0 & 1 & 0 & \frac{p_f}{E_f+m} \end{pmatrix} \tag{A.26}$$

$$u_i^{\mathrm{T}}(p_i) = \begin{pmatrix} 0 & 1 & 0 & \frac{-p_i}{E_i+m} \end{pmatrix}. \tag{A.27}$$

With help of the integral formula [10]

$$\int_{-\infty}^{\infty} \mathrm{d}\xi\, H_n(\xi)\exp[-(\xi-\alpha)^2] = \sqrt{\pi}(2\alpha)^n \tag{A.28}$$

and with the substitutions $k_{fy} = a_f - q_y$ and $k_{iy} = q_y - a_i$ which are implied in the δ functions in the first term in (A.23), the integration

$$I_1 = \int \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{e}^{-\mathrm{i}k_{fx}x_1} I_0(x_1,a_f)S_n(x_1,x_2,\boldsymbol{q})\mathrm{e}^{\mathrm{i}k_{ix}x_2} I_0(x_2,a_i) \tag{A.29}$$

can be carried out to give

$$I_1 = \frac{(\lambda^3 k_f^+ k_i^-)^{(n-1)}}{2^n n!} A_n \exp\left[-\frac{\lambda^2}{4}(k_i^2\sin^2\theta_i + k_f^2\sin^2\theta_f)\right] \\ \times \exp[\mathrm{i}\lambda^2(a_f k_{fx} + \tfrac{1}{2}k_{fx}k_{fy}) - \mathrm{i}\lambda^2(a_i k_{ix} + \tfrac{1}{2}k_{ix}k_{iy})] \tag{A.30}$$

where $k_i^{\pm} = k_{ix} \pm \mathrm{i}k_{iy}$ and $k_f^{\pm} = k_{fx} \pm \mathrm{i}k_{fy}$ and A_n is defined by

$$A_n = \begin{pmatrix} 2n(q_0+m) & 0 & -2nq_z & -2nk_i^- \\ 0 & (q_0+m)\lambda^2 k_f^+ k_i^- & -2nk_f^+ & q_z\lambda^2 k_f^+ k_i^- \\ 2nq_z & 2nk_i^- & -2n(q_0-m) & 0 \\ 2nk_f^+ & -q_z\lambda^2 k_f^+ k_i^- & 0 & -(q_0-m)\lambda^2 k_f^+ k_i^- \end{pmatrix}. \tag{A.31}$$

Similarly the integration in the second term in (A.23)

$$I_2 = \int \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{e}^{\mathrm{i}k_{ix}x_1} I_0(x_1,a_f)S_n(x_1,x_2,\boldsymbol{q})\mathrm{e}^{-\mathrm{i}k_{fx}x_2} I_0(x_2,a_i) \tag{A.32}$$

can be performed to yield

$$I_2 = \frac{(\lambda^3 k_f^+ k_i^-)^{(n-1)}}{2^n n!} B_n \exp\left[-\frac{\lambda^2}{4}(k_i^2\sin^2\theta_i + k_f^2\sin^2\theta_f)\right] \\ \times \exp[-\mathrm{i}\lambda^2(a_f k_{ix} - \tfrac{1}{2}k_{ix}k_{iy}) + \mathrm{i}\lambda^2(a_i k_{fx} + \tfrac{1}{2}k_{fx}k_{fy})] \tag{A.33}$$

with the matrix B_n given by

$$B_n = \begin{pmatrix} 2n(q_0+m) & 0 & -2nq_z & 2nk_f^- \\ 0 & (q_0+m)\lambda^2 k_f^- k_i^+ & 2nk_i^+ & q_z\lambda^2 k_f^- k_i^+ \\ 2nq_z & -2nk_f^- & -2n(q_0-m) & 0 \\ -2nk_i^+ & -q_z\lambda^2 k_f^- k_i^+ & 0 & -(q_0-m)\lambda^2 k_f^- k_i^+ \end{pmatrix}. \tag{A.34}$$

After the integrations over x_1 and x_2, two scalar matrix products, $P_1 = \bar{u}_f(p_f)\hat{e}_f A_n \hat{e}_i u_i(p_i)$ and $P_2 = \bar{u}_f(p_f)\hat{e}_i B_n \hat{e}_f u_i(p_i)$, are left, which can be easily calculated to give

$$P_1 = 2n\left[\frac{(q_0+m)p_i p_f}{(E_f+m)(E_i+m)} + (q_0-m)\right] e_f^+ e_i^- \\ +2n\left[\frac{p_f}{E_f+m} + \frac{p_i}{E_i+m}\right](k_i^- e_f^+ e_{iz} + k_f^+ e_{fz} e_i^- - q_z e_f^+ e_i^-) \\ +\lambda^2 k_f^+ k_i^- \left[\frac{(q_0+m)p_i p_f}{(E_f+m)(E_i+m)} + (q_0-m) + \frac{q_z p_f}{E_f+m} + \frac{q_z p_i}{E_i+m}\right] e_{fz} e_{iz} \tag{A.35}$$

$$P_2 = 2n\left[\frac{(q_0+m)p_ip_f}{(E_f+m)(E_i+m)} + (q_0-m)\right]e_i^+e_f^-$$
$$-2n\left[\frac{p_f}{E_f+m} + \frac{p_i}{E_i+m}\right](k_i^+e_f^-e_{iz} + k_f^-e_i^-e_{fz} + q_ze_i^+e_f^-)$$
$$+\lambda^2k_f^+k_i^-\left[\frac{(q_0+m)p_ip_f}{(E_f+m)(E_i+m)} + (q_0-m) + \frac{q_zp_f}{E_f+m} + \frac{q_zp_i}{E_i+m}\right]e_{fz}e_{iz}. \tag{A.36}$$

After substituting (A.35) and (A.36) into (A.23) and integrating over q_y, q_z and q_0, the scattering amplitude (5) is derived.

References

[1] Herold H 1979 *Phys. Rev.* D **19** 2868
[2] Dermer C D 1990 *APJ* **360** 197
[3] Sturner S J 1995 *APJ* **446** 292
[4] Blumenthal G R and Gould R J 1970 *Rev. Mod. Phys.* **42** 237
[5] Conthier P L *et al* 2000 *Preprint* astro-ph/0005072
[6] Xu H H and Tsai C H 1990 *Phys. Rev.* A **41** 53
[7] Daugherty J K and Vantura J 1978 *Phys. Rev.* D **18** 1053
[8] Daugherty J K and Harding A K 1989 *APJ* **336** 861
[9] Andersen J O and Haugset T 1995 *Phys. Rev.* D **51** 3073
[10] Gradshteyn I S and Ryzhik I M 1980 *Table of Integrals, Series, and Products* (New York: Academic)